Some finite semigroups and semirings without finite identity basis

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(with Yuzhu Chen, Xun Hu, Yanfeng Luo, and Miaomiao Ren)

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The Finite Basis Problem (FBP) is natural by itself, but it has also revealed a number of interesting and unexpected relations to many issues of theoretical and practical importance ranging from feasible algorithms for membership in certain classes of formal languages to classical number-theoretic conjectures such as the Twin Prime, Goldbach, existence of odd perfect numbers and the infinitude of even perfect numbers.
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The algorithmic version of the FBP for the class of finite algebras is known as Tarski’s problem.

For general algebras (and even for groupoids) the corresponding problem is shown by Ralph McKenzie to be undecidable. For semigroups it is still open and attracts lots of attention. See my survey “The finite basis problem for finite semigroups”, Sci. Math. Jap., Vol. 53, 171–199, 2001; its (occasionally) updated version is available through my webpage: http://csseminar.kadm.usu.ru/volkov

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For instance, the direct product of two finitely based finite semigroups can be non-finitely based. An old example: Let $A_2 = \langle a, b \mid aba = a^2 = a, \ bab = b, \ b^2 = 0 \rangle$. The semigroup $A_2$ consists of 5 elements and can be thought of the semigroup formed by the following $2 \times 2$-matrices:

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}.$$

The semigroup $A_2$ is finitely based (Trahtman, 1981) while its direct product with any non-trivial finite group is non-finitely based ($\sim$, 1989).
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One can try to use examples like this in order to attack Tarski’s Problem for semigroups. Does the class $FBA_2$ of finite semigroups $S$ such that $A_2 \times S$ is finitely based have decidable membership? A negative answer to this question would imply a negative solution to Tarski’s Problem for semigroups. This is probably too much to hope. But if one proves that the membership problem for $FBA_2$ is hard (say, NP-hard), one can deduce that so is Tarski’s Problem.
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Our Result

We have described semigroups in $FBA_2$ which have central idempotents.

Recall that an element $e$ of a semigroup $S$ is called an idempotent if $e^2 = e$ and is said to be central if $es = se$ for all $s \in S$. 
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Mikhail Volkov et al
Non-finitely based finite semigroups
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**Theorem (Chen, Hu, Luo, ∼, 2016)**

A finite semigroup $S$ with central idempotents is such that the direct product $A_2 \times S$ is finitely based if and only if $S$ equationally equivalent to a direct product of a finite nilpotent semigroup with a semilattice.
A semiring is an algebra of the form \((S, +, \cdot)\) such that
- both \((S, +)\) and \((S, \cdot)\) are semigroups, and
- multiplication distributes over addition.

A very important special case is the one of additively idempotent semirings (ai-semirings for short) in which the laws \(a + a = a\) and \(a + b = b + a\) hold.
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**Theorem (Ren, $\sim$, 2016)**

- There is a 28-element ai-semiring satisfying $x^4 \preceq x$ which is non-finitely based.
- There is a 9-element ai-semiring satisfying $x^5 \preceq x$ which is non-finitely based.
The proof relies on the properties of flat extensions of groups. If $G$ is a group, its flat extension $G^\flat$ is the ai-semiring on the set $G \cup \{0\}$ (where 0 is a new symbol) with the addition defined by $g + g = g$ for every $g \in G \cup \{0\}$ and $g + h = 0$ for all different $g, h \in G \cup \{0\}$.

The multiplication in $G^\flat$ extends the multiplication in $G$ and satisfies $g0 = 0g = 0$ for every $g \in G \cup \{0\}$.

Key observation: If $G$ has no finite basis of quasiidentities, then $G^\flat$ has no finite identity basis.

This is a special case of a deep general result by Marcel Jackson (Flat algebras and the translation of universal Horn logic to equational logic, J. Symbolic Logic, 73, no.1 (2008) 90–128).
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