Axiomatization of if-then-else over $C$-algebras

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IIT Guwahati

February 11, 2017
if-then-else

if A then f else g
if-then-else

1. McCarthy (1963)
2. Igarashi (1971)
3. Sethi (1978)
5. Pigozzi (1991)
**Definition**

Let \( \langle B, \lor, \land, \neg, T, F \rangle \) be a Boolean algebra. Let \( S \) be a set. A **B-set** is defined to be a pair \((S, B)\), such that there is a **B-action** \( B \times S \times S \to S \), denoted by \( \alpha[a, b] \), satisfying for all \( \alpha, \beta \in B \) and \( a, b, c \in S \),
Definition
Let \( \langle B, \lor, \land, \neg, T, F \rangle \) be a Boolean algebra. Let \( S \) be a set. A \textit{B-set} is defined to be a pair \((S, B)\), such that there is a \textit{B-action} \( B \times S \times S \to S \), denoted by \( \alpha[a, b] \), satisfying for all \( \alpha, \beta \in B \) and \( a, b, c \in S \),

\[
\begin{align*}
\alpha[a, a] &= a \\
\alpha[\alpha[a, b], c] &= \alpha[a, c] \\
\alpha[a, \alpha[b, c]] &= \alpha[a, c] \\
F[a, b] &= b \\
\neg\alpha[a, b] &= \alpha[b, a] \\
(\alpha \land \beta)[a, b] &= \alpha[\beta[a, b], b]
\end{align*}
\]
Example

$(S, 2)$ is always a $B$-set, for any set $S$. Such a pair will be called a basic $B$-set.

\[
T[a, b] = a; \\
F[a, b] = b.
\]
Example: $(T(X), 2^X)$ is a $B$-set where $T(X)$ denotes the set of all total functions over $X$.

Theorem ([2]): Every $B$-set is a subdirect product of basic $B$-sets.
Example

\((T(X), 2^X)\) is a \(B\)-set where \(T(X)\) denotes the set of all total functions over \(X\).

\[
\alpha [g, h](x) = \begin{cases} 
    g(x), & \text{if } x \in \alpha; \\
    h(x), & \text{otherwise.}
\end{cases}
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Theorem ([2])

Every \(B\)-set is a subdirect product of basic \(B\)-sets.
Agreeable $B$-sets

Definition

A $B$-set $(S, B)$ equipped with an operation $\ast$:

\[ S \times S \rightarrow B \]

is said to be agreeable if it satisfies the following axioms for all $s, t, u, v \in S$ and $\alpha \in B$:

1. \( s \ast s = T \) \hspace{1cm} (7)
2. \( (s \ast t)[s, t] = t \) \hspace{1cm} (8)
3. \( \alpha[s, t] \ast \alpha[u, v] = \alpha[s \ast u, t \ast v] \) \hspace{1cm} (9)
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$$\alpha[s, t] \ast \alpha[u, v] = \alpha[s \ast u, t \ast v]$$  \hspace{1cm} (9)
Let $S$ be any set. The pair $(S, 2)$ is an agreeable $B$-set.

$$s \ast t = \begin{cases} T, & \text{if } s = t; \\ F, & \text{otherwise}. \end{cases}$$

These $B$-sets are called basic agreeable $B$-sets.

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Every agreeable $B$-set is a subdirect product of basic agreeable $B$-sets.
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**Example**

Let $S$ be any set. The pair $(S, 2)$ is an agreeable $B$-set.

$$s \ast t = \begin{cases} 
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**Theorem ([2])**

*Every agreeable $B$-set is a subdirect product of basic agreeable $B$-sets.*
The algebra of conditional logic
The algebra of conditional logic

Definition

A *C-algebra* is an algebra \( \langle M, \lor, \land, \neg \rangle \) of type \((2, 2, 1)\), which satisfies the following axioms for all \( \alpha, \beta, \gamma \in M \):

\[
\neg \neg \alpha = \alpha \quad (10)
\]
\[
\neg (\alpha \land \beta) = \neg \alpha \lor \neg \beta \quad (11)
\]
\[
(\alpha \land \beta) \land \gamma = \alpha \land (\beta \land \gamma) \quad (12)
\]
\[
\alpha \land (\beta \lor \gamma) = (\alpha \land \beta) \lor (\alpha \land \gamma) \quad (13)
\]
\[
(\alpha \lor \beta) \land \gamma = (\alpha \land \gamma) \lor (\neg \alpha \land \beta \land \gamma) \quad (14)
\]
\[
\alpha \lor (\alpha \land \beta) = \alpha \quad (15)
\]
\[
(\alpha \land \beta) \lor (\beta \land \alpha) = (\beta \land \alpha) \lor (\alpha \land \beta) \quad (16)
\]
Example

Any Boolean algebra, $B$ is a $C$-algebra, by definition. In particular, the two-element Boolean algebra, $2$ is a $C$-algebra.

Example

The following is an example of a three-element $C$-algebra, $3 = \{ T, F, U \}$.

<table>
<thead>
<tr>
<th>$\neg$</th>
<th>$\land$</th>
<th>$T$</th>
<th>$F$</th>
<th>$U$</th>
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We will say that a $C$-algebra $M$ has $T, F, U$ if it has an identity for $\land$, an identity for $\lor$ and a fixed point for $\neg$. 
The algebra of disjoint alternatives
The algebra of disjoint alternatives

Definition
An ada is a $C$-algebra $M$ equipped with an additional unary operation $(\ )\downarrow$ subject to the following equations for all $\alpha, \beta \in M$:

$$F\downarrow = F \quad (17)$$
$$U\downarrow = F \quad (18)$$
$$T\downarrow = T \quad (19)$$
$$\alpha \wedge \beta\downarrow = \alpha \wedge (\alpha \wedge \beta)\downarrow \quad (20)$$
$$\alpha\downarrow \lor \neg(\alpha\downarrow) = T \quad (21)$$
$$\alpha = \alpha\downarrow \lor \alpha \quad (22)$$
Example

The three element $C$-algebra $3$ with the unary operation $(\ )\downarrow$ defined as follows forms an ada. This ada will hereafter be denoted by $3$.

\[ T\downarrow = T \]
\[ U\downarrow = F = F\downarrow \]
Example

The three element $C$-algebra $3$ with the unary operation $(\ )^\downarrow$ defined as follows forms an ada. This ada will hereafter be denoted by $3$.

\[
T^\downarrow = T
\]
\[
U^\downarrow = F = F^\downarrow
\]

- Every ada is a subalgebra of a product of copies of $3$. ([4]).
Example

The three element C-algebra 3 with the unary operation ( )↓ defined as follows forms an ada. This ada will hereafter be denoted by 3.

\[ T^{\downarrow} = T \]
\[ U^{\downarrow} = F = F^{\downarrow} \]

- Every ada is a subalgebra of a product of copies of 3. ([4]).
- Also \( 3^X \) is an ada with operations defined pointwise.
Example

The three element $C$-algebra $3$ with the unary operation $( )\downarrow$ defined as follows forms an ada. This ada will hereafter be denoted by $3$.

\[ T\downarrow = T \]
\[ U\downarrow = F = F\downarrow \]

- Every ada is a subalgebra of a product of copies of $3$. ([4]).
- Also $3^X$ is an ada with operations defined pointwise.
- Moreover $3$ is simple.
Definition

**C-sets**

Let $X$ be a set and $\bot \not\in X$. The pointed set $X \cup \{\bot\}$ with base point $\bot$ is denoted by $X_\bot$. The set of all total functions over $X_\bot$ which fix $\bot$ is denoted by $T_0(X_\bot)$, i.e. $T_0(X_\bot) = \{f \in T(X_\bot) : f(\bot) = \bot\}$.

Let $S_\bot$ be a pointed set with base point $\bot$ and $M$ be a $C$-algebra with $T, F, U$. The pair $(S_\bot, M)$ equipped with an action $\lbrack , \rbrack : M \times S_\bot \times S_\bot \to S_\bot$ is called a $C$-set if it satisfies the following axioms for all $\alpha, \beta \in M$ and $s, t, u, v \in S_\bot$: AAA93: Axiomatization of if-then-else over $C$-algebras
**C-sets**

**Notation**

Let $X$ be a set and $\bot \notin X$. The pointed set $X \cup \{\bot\}$ with base point $\bot$ is denoted by $X_\bot$. The set of all total functions over $X_\bot$ which fix $\bot$ is denoted by $T_o(X_\bot)$, i.e. $T_o(X_\bot) = \{f \in T(X_\bot) : f(\bot) = \bot\}$. 
C-sets

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Let $X$ be a set and $\bot \notin X$. The pointed set $X \cup \{\bot\}$ with base point $\bot$ is denoted by $X_\bot$. The set of all total functions over $X_\bot$ which fix $\bot$ is denoted by $T_o(X_\bot)$, i.e.

$$T_o(X_\bot) = \{ f \in T(X_\bot) : f(\bot) = \bot \}.$$ 

Definition

Let $S_\bot$ be a pointed set with base point $\bot$ and $M$ be a C-algebra with $T, F, U$. The pair $(S_\bot, M)$ equipped with an action

$$[-, -] : M \times S_\bot \times S_\bot \to S_\bot$$

is called a C-set if it satisfies the following axioms for all $\alpha, \beta \in M$ and $s, t, u, v \in S_\bot$: 
### References

AAA93: Axiomatization of if-then-else over $C$-algebras
**Definition**

\[ U[s, t] = \bot \]  \hspace{1cm} (23)
\[ F[s, t] = t \]  \hspace{1cm} (24)
\[ (\neg \alpha)[s, t] = \alpha[t, s] \]  \hspace{1cm} (25)
\[ \alpha[\alpha[s, t], u] = \alpha[s, u] \]  \hspace{1cm} (26)
\[ \alpha[s, \alpha[t, u]] = \alpha[s, u] \]  \hspace{1cm} (27)
\[ (\alpha \land \beta)[s, t] = \alpha[\beta[s, t], t] \]  \hspace{1cm} (28)
\[ \alpha[\beta[s, t], \beta[u, v]] = \beta[\alpha[s, u], \alpha[t, v]] \]  \hspace{1cm} (29)
\[ \alpha[s, t] = \alpha[t, t] \Rightarrow (\alpha \land \beta)[s, t] = (\alpha \land \beta)[t, t] \]  \hspace{1cm} (30)
Example

Let $S_{\perp}$ be a pointed set with base point $\perp$. The pair $(S_{\perp}, 3)$ is a $C$-set with respect to the following action for all $a, b \in S_{\perp}$ and $\alpha \in 3$:

$$\alpha \left[ a, b \right] = \begin{cases} a, & \text{if } \alpha = T; \\ b, & \text{if } \alpha = F; \\ \perp, & \text{if } \alpha = U. \end{cases}$$

These $C$-sets will be called basic $C$-sets.

Example

Let $M$ be a $C$-algebra with $T, F, U$. By treating $M$ as a pointed set with base point $U$, the pair $(M, M)$ is a $C$-set under the following action for all $\alpha, \beta, \gamma \in M$:

$$\alpha \wedge \beta, \gamma = (\alpha \wedge \beta) \vee (\neg \alpha \wedge \gamma).$$
Example

Let $S_\bot$ be a pointed set with base point $\bot$. The pair $(S_\bot, 3)$ is a C-set with respect to the following action for all $a, b \in S_\bot$ and $\alpha \in 3$:

$$\alpha[a, b] = \begin{cases} 
a, & \text{if } \alpha = T; 

b, & \text{if } \alpha = F; 

\bot, & \text{if } \alpha = U.
\end{cases}$$

These C-sets will be called basic C-sets.
Example

Let $S_{\bot}$ be a pointed set with base point $\bot$. The pair $(S_{\bot}, 3)$ is a $C$-set with respect to the following action for all $a, b \in S_{\bot}$ and $\alpha \in 3$:

$$
\alpha[a, b] = \begin{cases} 
a, & \text{if } \alpha = T; 
b, & \text{if } \alpha = F; 
\bot, & \text{if } \alpha = U.
\end{cases}
$$

These $C$-sets will be called **basic $C$-sets**.

Example

Let $M$ be a $C$-algebra with $T, F, U$. By treating $M$ as a pointed set with base point $U$, the pair $(M, M)$ is a $C$-set under the following action for all $\alpha, \beta, \gamma \in M$:

$$
\alpha \llbracket \beta, \gamma \rrbracket = (\alpha \land \beta) \lor (\neg \alpha \land \gamma).
$$
### Examples

Consider $T \otimes (X \perp)$ as a pointed set with base point $\perp$, the constant function taking the value $\perp$. The pair $(T \otimes (X \perp), 3X)$ is a $C$-set with the following action. For all $f, g \in T \otimes (X \perp)$ and $\alpha \in 3X$,

$$\alpha[f, g](x) =
\begin{cases}
  f(x), & \text{if } \alpha(x) = T; \\
  g(x), & \text{if } \alpha(x) = F; \\
  \perp, & \text{otherwise}.
\end{cases}$$

(31)
Example

Consider $T_o(X_\bot)$ as a pointed set with base point $\zeta_\bot$, the constant function taking the value $\bot$. The pair $(T_o(X_\bot), 3^X)$ is a $C$-set with the following action. For all $f, g \in T_o(X_\bot)$ and $\alpha \in 3^X$,

$$\alpha[f, g](x) = \begin{cases} 
 f(x), & \text{if } \alpha(x) = T; \\
 g(x), & \text{if } \alpha(x) = F; \\
 \bot, & \text{otherwise.} 
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Example
Consider \( S^X_{\perp} \), the set of all functions from \( X \) to \( S_{\perp} \), as a pointed set with base point \( \zeta_{\perp} \). The pair \( (S^X_{\perp}, 3^X) \) is a C-set under the action given in (31).

Example
Consider \( \mathcal{T}(X_{\perp}) \), the set of all total functions on \( X_{\perp} \), as a pointed set with base point \( \zeta_{\perp} \). The pair \( (\mathcal{T}(X_{\perp}), 3^X) \) is a C-set under the action given in (31), where \( f, g \in \mathcal{T}(X_{\perp}) \) and \( \alpha \in 3^X \).
**C-set properties**

(i) \( \alpha[\bot, \bot] = \bot \).

(ii) If \( \alpha[s, u] = \alpha[r, r] \) for some \( u \in S_\bot \) then \( \alpha[s, r] = \alpha[r, r] \).

(iii) If \( \alpha[s, u] = \alpha[t, u] \) for some \( u \in S_\bot \) then \( \alpha[s, v] = \alpha[t, v] \) for all \( v \in S_\bot \).

(iv) If \( \alpha[s, t] = \alpha[t, t] \) then \( (\beta \land \alpha)[s, t] = (\beta \land \alpha)[t, t] \).

(v) For each \( \alpha \in M_\# \) and \( s \in S_\bot \), we have \( \alpha[s, s] = s \).

**Corollary**

The pair \((S_\bot, M_\#)\) is a \( B \)-set.
C-set properties

Proposition

The following statements hold for all $\alpha, \beta \in M$ and $s, t, r \in S_{\bot}$.

(i) $\alpha[\bot, \bot] = \bot$.

(ii) If $\alpha[s, u] = \alpha[r, r]$ for some $u \in S_{\bot}$ then $\alpha[s, r] = \alpha[r, r]$.

(iii) If $\alpha[s, u] = \alpha[t, u]$ for some $u \in S_{\bot}$ then $\alpha[s, v] = \alpha[t, v]$ for all $v \in S_{\bot}$.

(iv) If $\alpha[s, t] = \alpha[t, t]$ then $(\beta \land \alpha)[s, t] = (\beta \land \alpha)[t, t]$.

(v) For each $\alpha \in M_{\#}$ and $s \in S_{\bot}$, we have $\alpha[s, s] = s$. 

C-set properties

Proposition

The following statements hold for all $\alpha, \beta \in M$ and $s, t, r \in S\bot$.

(i) $\alpha[\bot, \bot] = \bot$.

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(iii) If $\alpha[s, u] = \alpha[t, u]$ for some $u \in S\bot$ then $\alpha[s, v] = \alpha[t, v]$ for all $v \in S\bot$.

(iv) If $\alpha[s, t] = \alpha[t, t]$ then $(\beta \land \alpha)[s, t] = (\beta \land \alpha)[t, t]$.

(v) For each $\alpha \in M\#_\#$ and $s \in S\bot$, we have $\alpha[s, s] = s$.

Corollary

The pair $(S\bot, M\#_\#)$ is a $B$-set.
Representation of a subclass of $C$-sets
Representation of a subclass of $C$-sets

**Definition**

A *congruence* on a $C$-set is a pair $(\sigma, \tau)$, where $\sigma$ is an equivalence relation on $S_\bot$ and $\tau$ is a congruence on the ada $M$ such that

$$(s, t), (u, v) \in \sigma, (\alpha, \beta) \in \tau \Rightarrow (\alpha[s, u], \beta[t, v]) \in \sigma.$$
Representation of a subclass of $C$-sets

**Definition**

A *congruence* on a $C$-set is a pair $(\sigma, \tau)$, where $\sigma$ is an equivalence relation on $S_{\perp}$ and $\tau$ is a congruence on the ada $M$ such that

$$(s, t), (u, v) \in \sigma, (\alpha, \beta) \in \tau \Rightarrow (\alpha[s, u], \beta[t, v]) \in \sigma.$$ 

**Definition**

For each maximal congruence $\theta$ on $M$, we define a relation on $S_{\perp}$ by

$$E_\theta = \{(s, t) \in S_{\perp} \times S_{\perp} : \beta[s, t] = \beta[t, t] \text{ for some } \beta \in \overline{T}_\theta\}.$$
Lemma
The relation $E_{\theta}$ is an equivalence on $S_{\perp}$.

Proposition
For any $\alpha \in M$, $\beta = \neg (\alpha \downarrow \lor (\neg \alpha \downarrow)) \lor U$ satisfies $\beta \land \alpha = U$.

Moreover, if $(\alpha, U) \in \theta$ then $(\beta, T) \in \theta$.

Proposition
For each $\alpha \in M$ and each $s, t \in S_{\perp}$, we have the following:

(i) $(\alpha, T) \in \theta \Rightarrow (\alpha \downarrow s, t) \in E_{\theta}$.

(ii) $(\alpha, F) \in \theta \Rightarrow (\alpha \downarrow s, t) \in E_{\theta}$.

(iii) $(\alpha, U) \in \theta \Rightarrow (\alpha \downarrow s, t) \in E_{\theta}$.
Lemma

The relation $E_{\theta}$ is an equivalence on $S_{\perp}$.
Lemma

The relation $E_\theta$ is an equivalence on $S_\bot$.

Proposition

For any $\alpha \in M$, $\beta = \neg(\alpha^\bot \lor (\neg\alpha)^\bot) \lor U$ satisfies $\beta \land \alpha = U$. Moreover, if $(\alpha, U) \in \theta$ then $(\beta, T) \in \theta$. 

AAA93: Axiomatization of if-then-else over C-algebras

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Lemma

The relation $E_\theta$ is an equivalence on $S_\bot$.

Proposition

For any $\alpha \in M$, $\beta = \neg(\alpha \downarrow \lor (\neg\alpha) \downarrow) \lor U$ satisfies $\beta \land \alpha = U$. Moreover, if $(\alpha, U) \in \theta$ then $(\beta, T) \in \theta$.

Proposition

For each $\alpha \in M$ and each $s, t \in S_\bot$, we have the following:

(i) $(\alpha, T) \in \theta \Rightarrow (\alpha[s, t], s) \in E_\theta$.

(ii) $(\alpha, F) \in \theta \Rightarrow (\alpha[s, t], t) \in E_\theta$.

(iii) $(\alpha, U) \in \theta \Rightarrow (\alpha[s, t], \bot) \in E_\theta$. 
Lemma
The pair \((E, \theta)\) is a \(C\)-set congruence.

Remark
Note that, as \(\theta\) is a maximal congruence on \(\text{ada} M\), \(M/\theta\) must be simple, i.e., \(M/\theta \cong \{3\}\). Further, the quotient set \(S/\theta\) can be treated as a pointed set with base point \(\bot\). Thus \((S/\theta, M/\theta)\) is a basic \(C\)-set under the action \(\alpha/\theta\) where

\[
\begin{cases}
    s/\theta, & \text{if } \alpha/\theta \in T/\theta; \\
    t/\theta, & \text{if } \alpha/\theta \in F/\theta; \\
    \bot/\theta, & \text{if } \alpha/\theta \in U/\theta.
\end{cases}
\]
Lemma

The pair \((E_\theta, \theta)\) is a C-set congruence.
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Remark

Note that, as \(\theta\) is a maximal congruence on \(\text{ada } M\), \(M/\theta\) must be simple, i.e., \(M/\theta \cong 3\). Further, the quotient set \(S_\bot/E_\theta\) can be treated as a pointed set with base point \(\bot\). Thus \((S_\bot/E_\theta, M/\theta)\) is a basic C-set under the action

\[
\overline{\alpha}^\theta [\overline{s}^{E_\theta}, \overline{t}^{E_\theta}] = \begin{cases} 
\overline{s}^{E_\theta}, & \text{if } \alpha \in \overline{T}^\theta; \\
\overline{t}^{E_\theta}, & \text{if } \alpha \in \overline{F}^\theta; \\
\bot^{E_\theta}, & \text{if } \alpha \in \overline{U}^\theta.
\end{cases}
\]
Lemma

In case of the $C$-set $(M, M)$, the equivalence $E_\theta$ on $M$, denoted by $E_{\theta_M}$, is a subset of $\theta$. 
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In case of the C-set \((M, M)\), the equivalence \(E_\theta\) on \(M\), denoted by \(E_{\theta M}\), is a subset of \(\theta\).

Lemma

\[ \bigcap_\theta E_\theta = \Delta_{S_{\perp}}, \] where \(\theta\) ranges over all maximal congruences on \(M\).
Lemma

In case of the C-set \((M, M)\), the equivalence \(E_\theta\) on \(M\), denoted by \(E_{\theta_M}\), is a subset of \(\theta\).

Lemma

\[ \bigcap \theta E_\theta = \Delta_{S_\perp} \], where \(\theta\) ranges over all maximal congruences on \(M\).

Remark

For \(\alpha, \beta \in M\) with \(\alpha \neq \beta\), let \(\theta_{\alpha, \beta}\) be a maximal congruence which separates \(\alpha\) and \(\beta\). Since \(\bigcap_{\alpha \neq \beta \in M} \theta_{\alpha, \beta} = \Delta_M\), the intersection of all maximal congruences on \(M\)

\[ \bigcap_{\theta \text{ maximal}} \theta = \Delta_M \]
<table>
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<tr>
<th>Background</th>
<th>C-sets</th>
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**Theorem**

Every C-set \((S \perp, M)\), where \(M\) is an ada, is a subdirect product of basic C-sets.

**Corollary**

An identity (quasi-identity) is satisfied in every C-set \((S \perp, M)\) where \(M\) is an ada if and only if it is satisfied in all basic C-sets.

AAA93: Axiomatization of if-then-else over C-algebras

IIT Guwahati
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Agreeable $C$-sets
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Boolean case:

$$(f \ast g)(x) = \begin{cases} T, & \text{if } f(x) = g(x); \\ F, & \text{otherwise.} \end{cases}$$
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If $f, g \in \mathcal{T}_o(X_\bot)$ then

$$(f \ast g)(x) = \begin{cases} T, & \text{if } f(x) = g(x) \text{ and } f(x) \neq \bot \neq g(x); \\ F, & \text{if } f(x) \neq g(x) \text{ and } f(x) \neq \bot \neq g(x); \\ U, & \text{otherwise.} \end{cases} \quad (32)$$
**Definition**

A C-set \((S_\bot, M)\) equipped with a function

\[ * : S_\bot \times S_\bot \rightarrow M \]

is said to be *agreeable* if it satisfies the following axioms for all \(s, t, u, v \in S_\bot\) and \(\alpha \in M\):

\[ (s * s)[s, \bot] = s \]  \hspace{1cm} (33)

\[ \bot * s = U = s * \bot \]  \hspace{1cm} (34)

\[ (s * t)[s, t] = (s * t)[t, t] \]  \hspace{1cm} (35)

\[ \alpha[s, t] * \alpha[u, v] = \alpha[s * u, t * v] \]  \hspace{1cm} (36)

\[ ((s * s = T) \land (s * t = U)) \Rightarrow t = \bot \]  \hspace{1cm} (37)
Every basic $C$-set is agreeable under the operation given by

$$s \ast t = \begin{cases} 
T, & \text{if } s = t (\neq \bot); \\
F, & \text{if } s \neq t (\neq \bot); \\
U, & \text{if } s = \bot \text{ or } t = \bot.
\end{cases}$$

(38)
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Every basic $C$-set is agreeable under the operation given by

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\]  

(38)

Proposition
The operation defined in (38) is the only possible operation under which a basic $C$-set can be made agreeable.
Example

The pair \((T_0(X⊥), 3X)\) is an agreeable \(C\)-set under the operation \(∗\) defined in (32).

Example

The \(C\)-set \((M, M)\) is agreeable under the operation \(\alpha ∗ β = (\alpha ∧ β) ∨ (\neg \alpha ∧ \neg β)\) which can be equivalently expressed in terms of the if-then-else action as \(\alpha ∗ β = \alpha J β, \neg β K\).
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The pair \((\mathcal{T}_o(X_\perp), 3^X)\) is an agreeable C-set under the operation \(*\) defined in (32).
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Example

The C-set \((M, M)\) is agreeable under the operation

\[
\alpha * \beta = (\alpha \land \beta) \lor (\neg \alpha \land \neg \beta)
\]

which can be equivalently expressed in terms of the if-then-else action as

\[
\alpha * \beta = \alpha[\beta, \neg \beta].
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Representation of a subclass of agreeable $C$-sets

Theorem
Every agreeable $C$-set $(S,\perp,M)$ where $M$ is an ada, is a subdirect product of agreeable basic $C$-sets.

Corollary
An identity (quasi-identity) is satisfied in every agreeable $C$-set $(S,\perp,M)$ where $M$ is an ada if and only if it is satisfied in all agreeable basic $C$-sets.

Corollary
In every agreeable $C$-set, where $M$ is an ada, we have $s^* t = t^* s$.

Proof method gives an alternate mechanism to prove the corresponding agreeable $B$-set representation theorem in [3].
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Thank you!
References


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