Concept Lattices of Relational Structures

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Mathematics
Relational Structure

\[
\begin{align*}
|\Delta| &= \{A, B, C, D, E\} \\
\sigma_\Delta &= \{B, C\} \\
\varphi_\Delta &= \{A, D, E\} \\
m_\Delta &= \{(A, B), (A, C)\} \\
f_\Delta &= \{(B, D), (B, E)\} \\
p_\Delta &= \{(A, B), (A, C), (B, D), (B, E)\}
\end{align*}
\]

Intension Graph

Power Context Family

\[
\begin{array}{c|c|c}
\sigma & \varphi \\
\hline
A & \times \\
B & \times \\
C & \\
D & \times \\
E & \times \\
\end{array}
\]

\[
\begin{array}{c|c|c|c}
 & m & f & p \\
\hline
(A,B) & \times & \times & \\
(A,C) & \times & \times & \\
(B,D) & \times & \times & \\
(B,E) & \times & \times & \\
\end{array}
\]
Primitive Positive Formulas

\[ \exists y_0 \exists y_1 \ p(y_0, y_1) \land p(y_1, x_0) \]

pp-Graph
Every pp-formula $\varphi(x_0, \ldots, x_{k-1})$ defines a $k$-ary relation $\varphi^\Delta$.

For every $k$-ary relation $A$ over $\Delta$, there is a strongest pp-formula $A^\Delta$ solved by all tuples in $A$.

A $k$-ary concept is a pair $(A, \varphi)$ with $A^\Delta = \varphi$ and $\varphi^\Delta = A$. 
Concept Lattice $\mathcal{B}_1(\Delta)$
family tree – unary concepts
Abstraction Operation: Direct Product of Graphs

\[
G_1 \times G_2 \xrightarrow{\pi_1} X \xrightarrow{\varphi} G_1 \times G_2 \xrightarrow{\pi_2} G_2
\]

\[
X\xrightarrow{\varphi_1} G_1 \quad \quad \quad X\xrightarrow{\varphi_2} G_2
\]
Philosophical Motivation
Starting Point: Modeling Concepts

- Is FCA really about modeling concepts?
- What is a concept?
Starting Point: Modeling Concepts

- Is FCA really about modeling concepts?
- What is a concept?
- How can we model concepts, if concepts don’t exist?
Learning a Concept

by example

by explanation

A sheep is a four-legged animal with wooly fur.
Significance of Primitive Positive Formulas

- Both data and intents are graphs
- Hypothesis: The basic concepts arise from observation (Positivism?)
- Thomas v. Aquin: "Nihil est in intellectu quod non prius fuerit in sensu"
Applications
Data Transformations

- Relational Database
- Object-oriented
- RDF/RDFS
- Power Context Family
- Concept Lattice
Example: Relational DB → PCF

### Person

<table>
<thead>
<tr>
<th>name</th>
<th>gender</th>
<th>parent</th>
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<tbody>
<tr>
<td>Anne</td>
<td>female</td>
<td></td>
</tr>
<tr>
<td>Bob</td>
<td>male</td>
<td>Anne</td>
</tr>
<tr>
<td>Chris</td>
<td>male</td>
<td>Anne</td>
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<tr>
<td>Dora</td>
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<td>Bob</td>
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<tr>
<td>Emily</td>
<td>female</td>
<td>Bob</td>
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### Concept

Lattices of Relational Structures

Mathematics

Philosophical Motivation

Applications

References

#### Example: Relational DB → PCF

**$\mathbb{K}_1$:**

<table>
<thead>
<tr>
<th></th>
<th>$\sigma$</th>
<th>$\varphi$</th>
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<tbody>
<tr>
<td>A</td>
<td></td>
<td>$\times$</td>
</tr>
<tr>
<td>B</td>
<td>$\times$</td>
<td></td>
</tr>
<tr>
<td>C</td>
<td>$\times$</td>
<td></td>
</tr>
<tr>
<td>D</td>
<td>$\times$</td>
<td></td>
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<tr>
<td>E</td>
<td>$\times$</td>
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**$\mathbb{K}_2$:**

<table>
<thead>
<tr>
<th></th>
<th>m</th>
<th>f</th>
<th>p</th>
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<tr>
<td>(A,B)</td>
<td>$\times$</td>
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<td></td>
</tr>
<tr>
<td>(A,C)</td>
<td>$\times$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>(B,D)</td>
<td></td>
<td>$\times$</td>
<td></td>
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<tr>
<td>(B,E)</td>
<td></td>
<td>$\times$</td>
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</table>
Example: Book Search

Author

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<thead>
<tr>
<th>name</th>
<th>nationality</th>
<th>date of birth</th>
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<tbody>
<tr>
<td>Lewis Carroll</td>
<td>British</td>
<td>1832-01-27</td>
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<tr>
<td>Virginia Woolf</td>
<td>British</td>
<td>1862-05-21</td>
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<td>Douglas Adams</td>
<td>British</td>
<td>1952-03-11</td>
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<td>J. K. Rowling</td>
<td>British</td>
<td>1965-07-31</td>
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<td>Stephen King</td>
<td>American</td>
<td>1947-08-21</td>
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<td>Dan Brown</td>
<td>American</td>
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Book

<table>
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<tr>
<th>title</th>
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<th>publication date</th>
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<td>Alice in Wonderland</td>
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<td>To the Lighthouse</td>
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<td>Harry Potter and the Deathly Hallows</td>
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<td>2007-07-21</td>
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<td>The Casual Vacancy</td>
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<td>2012-09-27</td>
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<td>The Shining</td>
<td>Stephen King</td>
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<td>Doctor Sleep</td>
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<td>The Da Vinci Code</td>
<td>Dan Brown</td>
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<td>Inferno</td>
<td>Dan Brown</td>
<td>2013-03-14</td>
</tr>
</tbody>
</table>

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References
Rudolf Wille
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